Review: Time–Dependent Maxwell's Equations

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

$$\nabla \times \vec{H}(t) = \frac{\partial \vec{D}(t)}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D}(t) = \rho$$

$$\nabla \cdot \vec{B}(t) = 0$$

$$\vec{D}(t) = \varepsilon \, \vec{E}(t)$$

$$\vec{B}(t) = \mu \, \vec{H}(t)$$

Electromagnetic quantities

Vector quantities in space

$\vec{E}$	Electric Field
$\vec{H}$	Magnetic Field
$\vec{D}$	Electric Flux (Displacement) Density
$\vec{B}$	Magnetic Flux (Induction) Density
$\vec{J}$	Current Density
$\dfrac{\partial \vec{D}}{\partial t}$	Displacement Current

ρ	Charge Density
ε	Dielectric Permittivity
μ	Magnetic Permeability

Material Constants

In free space:

$$\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12}\ [\mathrm{As/Vm}]\ \text{ or }\ [\mathrm{F/m}]$$

$$\mu = \mu_0 = 4\pi \times 10^{-7}\ [\mathrm{Vs/Am}]\ \text{ or }\ [\mathrm{Henry/m}]$$

In a material medium:

$$\varepsilon = \varepsilon_r\, \varepsilon_0 \quad ; \quad \mu = \mu_r\, \mu_0$$

ε_r = relative permittivity (dielectric constant)

μ_r = relative permeability

If the medium is anisotropic, the relative quantities are tensors:

$$\varepsilon_r = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad ; \quad \mu_r = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}$$

Electromagnetic Waves

For fast-varying phenomena, the displacement current cannot be neglected, and the full set of Maxwell's equations must be used

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}(t)}{dt}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}(t)}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Uniform Isotropic Media

The two curl equations are analogous to the coupled (first order) equations for voltage and current used in transmission lines. The solutions of this system of equations are waves. In order to obtain uncoupled (second order) equations we can operate with the curl once more. Under the assumption of uniform isotropic medium:

$$\nabla \times \nabla \times \vec{E}(t) = -\frac{\partial\left(\nabla \times \vec{B}(t)\right)}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}(t)$$

$$= -\mu \frac{\partial \vec{J}(t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

$$\nabla \times \nabla \times \vec{H}(t) = \nabla \times \vec{J} + \frac{\partial\left(\nabla \times \vec{D}(t)\right)}{\partial t} = \nabla \times \vec{J} + \varepsilon \frac{\partial}{\partial t} \nabla \times \vec{E}(t)$$

$$= \nabla \times \vec{J} - \varepsilon\mu \frac{\partial^2 \vec{H}(t)}{\partial t^2}$$

Generalized Wave Equation

From vector calculus, we also have

$$\nabla\times\nabla\times\vec{E}(t)=\nabla\nabla\cdot\vec{E}(t)-\nabla^2\vec{E}(t)$$

$$\nabla\times\nabla\times\vec{H}(t)=\nabla\underbrace{\nabla\cdot\vec{H}(t)}_{\boxed{\frac{1}{\mu}\nabla\cdot\vec{B}(t)=0}}-\nabla^2\vec{H}(t)=-\nabla^2\vec{H}(t)$$

Finally, we obtain the general wave equations

$$\nabla^2\vec{E}(t)-\nabla\nabla\cdot\vec{E}(t)-\mu\varepsilon\frac{\partial^2\vec{E}(t)}{\partial t^2}=\mu\frac{\partial\vec{J}(t)}{\partial t}$$

$$\nabla^2\vec{H}(t)-\mu\varepsilon\frac{\partial^2\vec{H}(t)}{\partial t}=-\nabla\times\vec{J}(t)$$

Free Space

In a region where the wave solution propagates away from charges and flowing currents, the wave equations can be simplified considerably. In such conditions, we have

$$\rho = 0 \quad \Rightarrow \quad \nabla \cdot \vec{E}(t) = \rho / \varepsilon = 0$$

$$\vec{J}(t) = 0$$

and the wave equations assume the familiar form

$$\nabla^2 \vec{E}(t) - \mu \, \varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{H}(t) - \mu \, \varepsilon \frac{\partial^2 \vec{H}(t)}{\partial t} = 0$$

Time Harmonic Fields

In engineering it is very important to consider time-harmonic fields with a sinusoidal time-variation. If we assume a steady-state situation (after all transients have died out) most physical situations may be investigated by considering one single frequency at a time.

This assumption leads to great simplifications in the algebra. It is also realistic, because in practical electromagnetics applications we often have a dominant frequency (carrier) to consider.

The time-harmonic fields have the form

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t + \varphi_E) \qquad \vec{H}(t) = \vec{H}_0 \cos(\omega t + \varphi_H)$$

We can use the complex phasor representation

$$\vec{E}(t) = \mathrm{Re}\left\{\vec{E}_0 e^{j\varphi_E} e^{j\omega t}\right\} \qquad \vec{H}(t) = \mathrm{Re}\left\{\vec{H}_0 e^{j\varphi_H} e^{j\omega t}\right\}$$

Phasors

We define

$$\vec{\mathrm{E}} = \vec{E}_0 e^{j\varphi_E} = \text{phasor of } \vec{E}(t)$$

$$\vec{\mathrm{H}} = \vec{H}_0 e^{j\varphi_H} = \text{phasor of } \vec{H}(t)$$

Maxwell's equations can be rewritten for phasors, with the time-derivatives transformed into linear terms

$$j\omega\vec{\mathrm{E}} = \text{phasor of } \frac{\partial \vec{E}(t)}{\partial t}$$

$$-\omega^2\vec{\mathrm{E}} = \text{phasor of } \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

Maxwell's Equations In Phasor Form

In phasor form, Maxwell's equations become

$$\boxed{\begin{aligned} &\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}} \\ &\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\varepsilon\,\vec{\mathbf{E}} \\ &\nabla \cdot \vec{\mathbf{D}} = \rho \\ &\nabla \cdot \vec{\mathbf{B}} = 0 \\ &\vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}} \\ &\vec{\mathbf{B}} = \mu\,\vec{\mathbf{H}} \end{aligned}}$$

$$\boxed{\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{v} \times \vec{\mathbf{B}}\right)}$$

where all electromagnetic quantities are phasors and functions only of space coordinates.

Helmholtz Equation

Let's consider first vacuum as a medium. The wave equations for phasors become Helmholtz equations

$$\boxed{\begin{aligned} \nabla^2 \vec{E} + \omega^2 \mu_0 \varepsilon_0 \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu_0 \varepsilon_0 \vec{H} = 0 \end{aligned}}$$

The general solutions for these differential equations are waves moving in 3-D space. Note, once again, that the two equations are uncoupled.

This means that each equation contains all the necessary information for the total electromagnetic field and one only needs to solve the equation for one field to completely specify the problem. The other field is obtained with a curl operation by invoking one of the original Maxwell equations.

Solution

This equation has a well known general solution

$$A \exp(-j\beta z) + B \exp(j\beta z)$$

where the propagation constant is

$$\beta = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c}$$

The wave that we have assumed is a plane wave and we have verified that it is a solution of Helmholtz equation. The general solution above has two possible components

$A \exp(-j\beta z)$	Forward wave, moving along positive z
$B \exp(j\beta z)$	Backward wave, moving along negative z

For the simple wave orientation chosen here, the problem is mathematically identical to the one solved earlier for voltage propagation in a homogeneous transmission line.

Forward Wave

If a specific electromagnetic wave is established in an infinite homogeneous medium, moving for instance along the positive direction, only the forward wave should be considered.

A reflected wave exists when a discontinuity takes place along the path of the forward wave (that is, the material medium changes properties, either abruprtly or gradually).

We can also assume that the amplitude of the forward plane wave solution is given and that it is in general a complex constant fixed by the conditions that generated the wave

$$A = E_0 \, e^{j\varphi}$$

We can write at last the phasor electric field describing a simple forward plane wave solution of Helmholtz equation as:

$$\vec{\mathrm{E}}_x(z) = E_0 \, e^{j\varphi} \, e^{-j\beta z} \, \hat{i}_x$$

In Time Domain

The corresponding time-dependent field is obtained by applying the inverse phasor transformation

$$\vec{E}_x(z,t) = \mathrm{Re}\left\{ \mathrm{E}_x(z)\, e^{j\omega t}\, \hat{i}_x \right\} = \mathrm{Re}\left\{ E_0\, e^{j\varphi}\, e^{-j\beta z}\, e^{j\omega t}\, \hat{i}_x \right\}$$

$$= E_0 \cos(\omega t - \beta z + \varphi)\, \hat{i}_x$$

The phasor magnetic field is obtained directly from the Maxwell equation for the electric field curl

$$\nabla \times \vec{\mathrm{E}} = \nabla \times \left(E_0\, e^{j\varphi}\, e^{-j\beta z}\, \hat{i}_x \right) = -j\omega\mu_0\, \vec{\mathrm{H}}$$

$$\vec{\mathrm{H}} = -\frac{\nabla \times \left(E_0\, e^{j\varphi}\, e^{-j\beta z}\, \hat{i}_x \right)}{j\omega\mu_0}$$

The Magnetic Field

The final result for the phasor magnetic field is

$$\vec{\mathbf{H}}_y(z) = -\frac{-j\beta\, E_0\, e^{j\varphi}\, e^{-j\beta z}}{j\omega\mu}\,\hat{i}_y =$$

$$= \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\,\mu_0}\, E_0\, e^{j\varphi}\, e^{-j\beta z}\,\hat{i}_y =$$

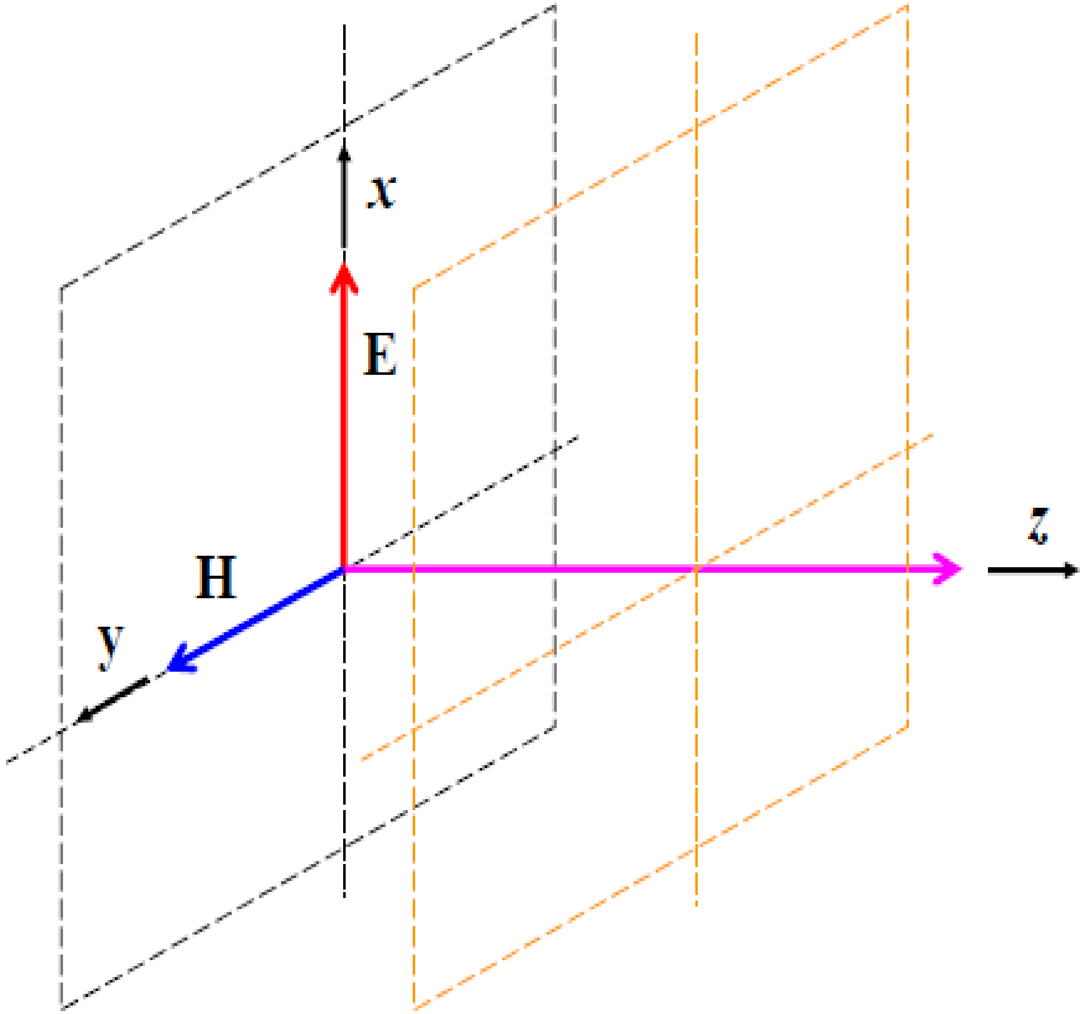
$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_0\, e^{j\varphi}\, e^{-j\beta z}\,\hat{i}_y = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathbf{E}_x(z)\,\hat{i}_y$$

We define

$$\boxed{\sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 \approx \mathbf{377}\,\boldsymbol{\Omega} \quad = \quad \text{Intrinsic impedance of vacuum}}$$

TEM Waves

We have found that the fields of the electromagnetic wave are perpendicular to each other, and that they are also perpendicular (or transverse) to the direction of propagation.



Power Flow

Electromagnetic power flows with the wave along the direction of propagation and it is also constant on the phase-planes. The power density is described by the time-dependent Poynting vector

$$\boxed{\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)}$$

The Poynting vector is perpendicular to both field components, and is parallel to the direction of wave propagation.

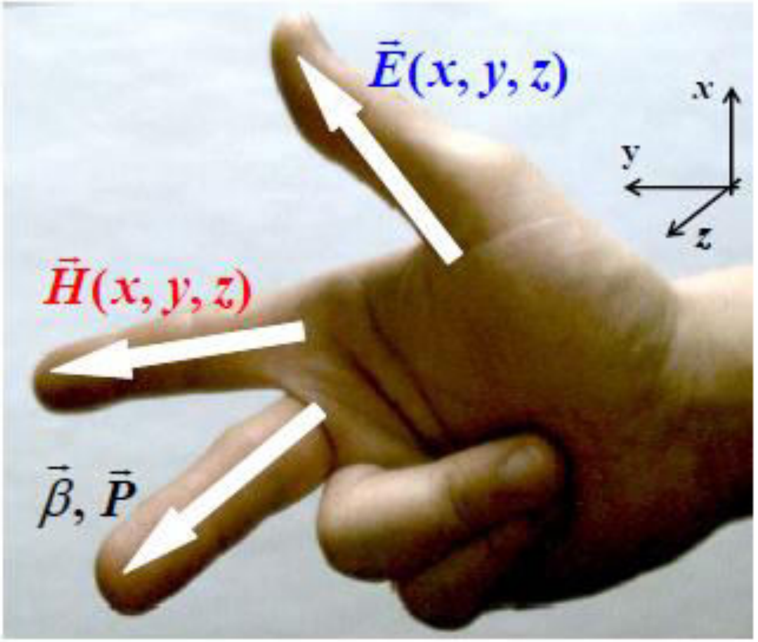
When the wave propagates on a general direction, which does not coincide with one of the cartesian axes, the propagation constant must be considered to be a vector with amplitude

$$\boxed{|\vec{\beta}| = \omega\sqrt{\mu\,\varepsilon}}$$

and direction parallel to the Poynting vector.

Direction of Propagation

The condition of mutual orthogonality between the field components and the Poynting vector is general and it applies to any plane wave with arbitrary direction of propagation. The mutual orientation chosen for the reference directions of the fields follows the right hand rule.



Periodic and Aperiodic Signals

So far, we have just verified that electromagnetic plane waves are possible solutions of the Maxwell equations for time-varying fields. One may wonder at this point if plane waves have practical physical relevance.

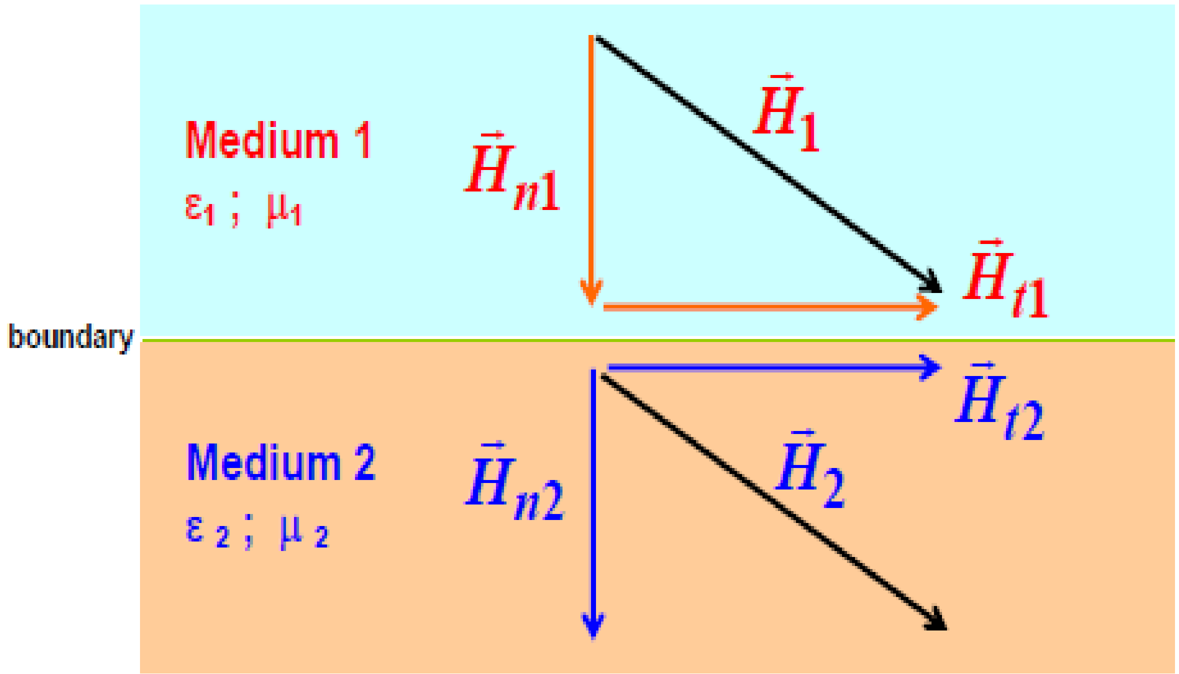
First of all, we should notice that plane waves are mathematically analogous to the exponential basis functions used in Fourier analysis. This means that a general wave, with more than one frequency component, can always be decomposed in terms of plane waves.

- For periodic signals, we have a discrete set of waves which are harmonics of the fundamental frequency (analogy with Fourier series).

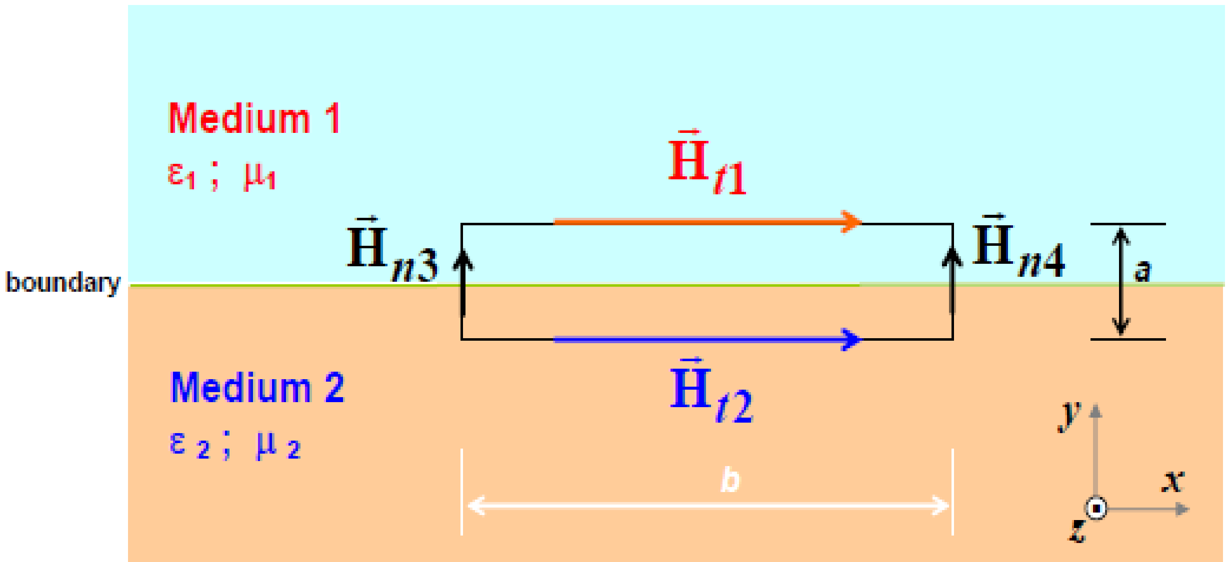
- For general signals, we must consider a continuum of frequencies in order to decompose in terms of elementary plane waves (analogy with Fourier transform).

Review of Boundary Conditions

Consider an electromagnetic field at the boundary between two materials with different properties. The tangent and the normal component of the fields must me examined separately, in order to understand the effects of the boundary.



Tangential Magnetic Field



Ampère's law for the boundary region in the figure can be written as

$$\nabla \times \vec{H} \Rightarrow \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + j\omega \varepsilon E_z$$

for materials with finite conductivity

$\Rightarrow H_{t2} - H_{t1} = 0$ Tangential components are conserved

for perfect conductors

$$\Rightarrow H_{t2} - H_{t1} = \lim_{a \to 0} (J_z a) = J_s \text{ (surface current)}$$

Summary

$$\boxed{\hat{n} \times (\vec{\mathrm{H}}_{t1} - \vec{\mathrm{H}}_{t2}) = \vec{J}_s}$$

$$\boxed{\hat{n} \times (\vec{\mathrm{E}}_{t1} - \vec{\mathrm{E}}_{t2}) = 0}$$

$$\boxed{\mathrm{D}_{1n} - \mathrm{D}_{2n} = \rho_s \qquad \mathrm{B}_{1n} - \mathrm{B}_{2n} = 0}$$

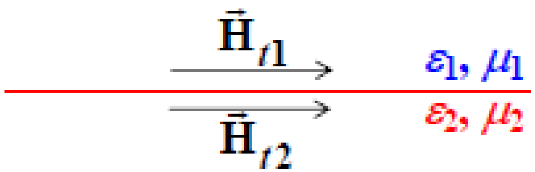
$\hat{n}$ = unit vector normal to the surface

n is a unit vector pointing from region 2 into region 1

Conditions on E and H

SUMMARY

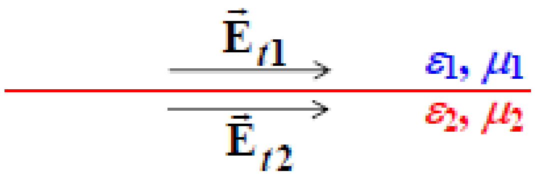
If medium 2 is perfect conductor



$$\vec{H}_{t1} = \vec{H}_{t2}$$

$$\hat{n} \times \vec{H}_{t1} = \vec{J}_s$$

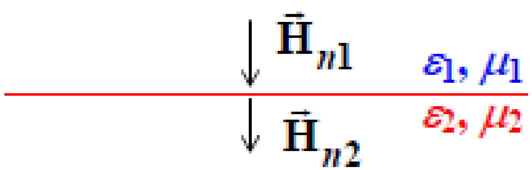
$$\vec{H}_{t2} = 0$$



$$\vec{E}_{t1} = \vec{E}_{t2}$$

$$\vec{E}_{t1} = 0$$

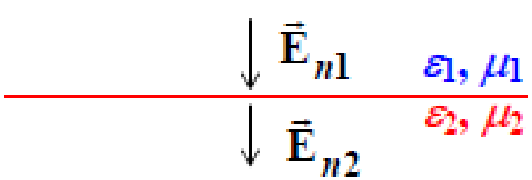
$$\vec{E}_{t2} = 0$$



$$\mu_1 \vec{H}_{n1} = \mu_2 \vec{H}_{n2}$$

$$\vec{H}_{n1} = 0$$

$$\vec{H}_{n2} = 0$$



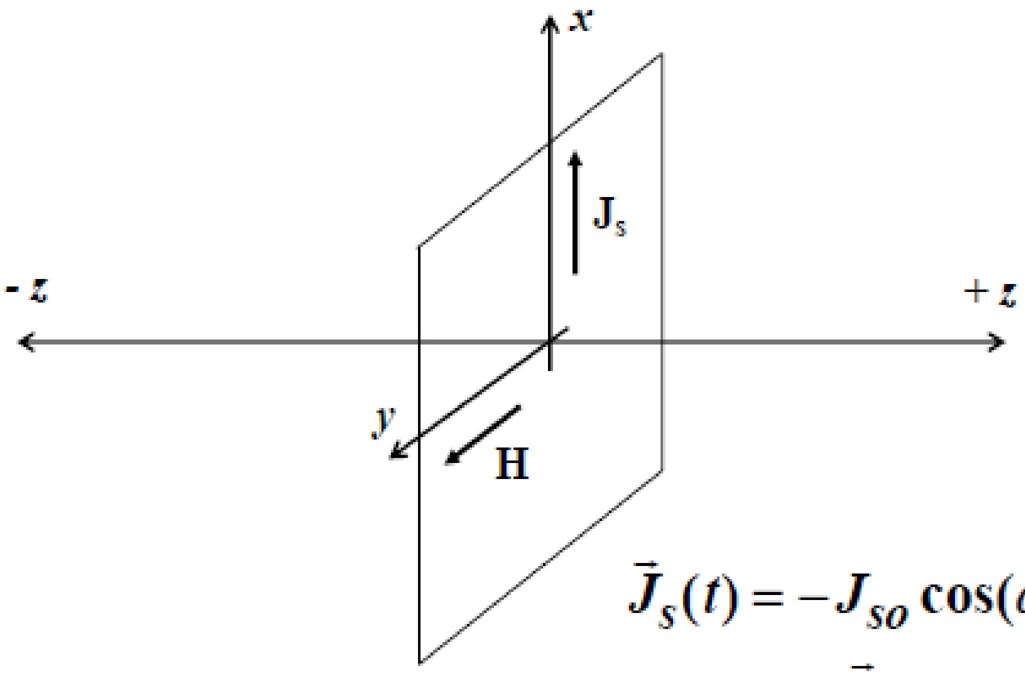
$$\varepsilon_1 \vec{E}_{n1} = \varepsilon_2 \vec{E}_{n2} + \rho_s$$

$$\vec{E}_{n1} = \rho_s / \varepsilon_1$$

$$\vec{E}_{n2} = 0$$

Example

An infinite current sheet generates a plane wave (free space on both sides)



$$\vec{J}_s(t) = -J_{so}\cos(\omega t)\,\hat{i}_x$$

$$\text{Phasor}\quad \vec{\mathrm{J}}_\mathrm{s} = -J_{so}\,\hat{i}_x$$

The E.M. field is transmitted on both sides of the infinitesimally thin sheet of current.

Solution

BOUNDARY CONDITIONS

$$\boxed{\hat{n} \times (\vec{\mathrm{H}}_{t1} - \vec{\mathrm{H}}_{t2}) = \vec{\mathrm{J}}_s}$$

$$\vec{\mathrm{H}}_{t1} - \vec{\mathrm{H}}_{t2} = J_{so}\hat{i}_x$$

$$\vec{\mathrm{E}}_{t1} = \vec{\mathrm{E}}_{t2}$$

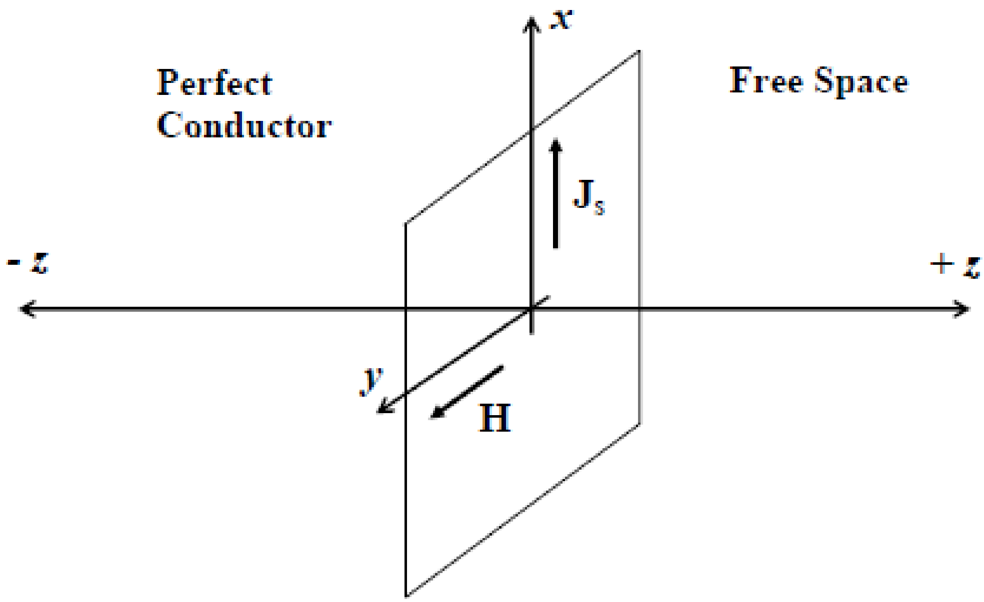
$$\left|\vec{\mathrm{E}}_{t1}\right| = \eta_0 \left|\vec{\mathrm{H}}_{t1}\right|$$

$$\text{Symmetry} \Rightarrow \left|\vec{\mathrm{H}}_{t1}\right| = \left|\vec{\mathrm{H}}_{t2}\right|$$

$$\mathrm{H}_1 = \frac{J_{so}}{2} \qquad \mathrm{H}_2 = -\frac{J_{so}}{2}$$

Example

A semi-infinite perfect conductor medium in contact with free space has uniform surface current and generates a plane wave



$$\vec{\mathbf{J}}_s = -J_{so}\cos(\omega t)\,\hat{i}_x$$

The E.M. field is zero inside the perfect conductor. The wave is only transmitted into free space.

Solution

BOUNDARY CONDITIONS

$$\boxed{\hat{n} \times (\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2}) = \vec{\mathbf{J}}_s}$$

$$\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2} = \vec{\mathbf{H}}_{t1} - 0 = J_{so}\,\hat{i}_x$$

$$\vec{\mathbf{E}}_{t2} = 0$$

$$\text{Asymmetry} \Rightarrow \left|\vec{\mathbf{H}}_{t1}\right| \neq \left|\vec{\mathbf{H}}_{t2}\right|$$

$$\mathbf{H}_{t1} = J_{so} \qquad \mathbf{H}_{t2} = 0$$